On a generalization of the pre-emptive resume priority

Philippe Nain

INRIA
CENTRE DE ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(3) 954 90 20

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Philippe NAIN
INRIA
Domaine de Voluceau - Rocquencourt
BP 105
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Abstract

Considered is a queueing system with two classes of customers and a single server, where the service policy is of threshold type. As soon as the amount of work required by the class 1 customers is greater than a fixed threshold, the class 1 customers get the server's attention ; otherwise the class 2 customers have the priority. Service interruptions can occur for both classes of customers on the basis of the above description of the service mechanism and in this case the service interruption discipline is Pre-emptive Resume Priority (PRP). This model, which turns out to be a generalization of the PRP queueing system, has potential applications in computer systems and in communication networks.

For Poisson inputs, exponential [resp. arbitrary] service time distribution for class 1 [resp. class 2] customers, we derive the Laplace-Stieltjes transform of the stationary joint distribution of the workload of the server, by reducing the analysis to the resolution of a boundary value problem. Explicit formulas are obtained.

Keywords
Queues ; Markov process ; Pre-emptive resume priority ; Riemann-Hilbert boundary value problem.

Résumé

Nous définissons et analysons une nouvelle politique de service de type "politique à seuil" pour un système de files d'attente à serveur unique et deux classes de clients. Le mécanisme de service est le suivant : dès que la demande de travail des clients de classe 1 excède un seuil fixé c ($c \in \mathbb{R}_+$), le serveur leur est alloué ; sinon il est alloué aux clients de classe 2. Ce mécanisme suppose implicitement que des interruptions de service peuvent survenir pour les deux classes de clients, et dans ce cas nous supposons que le service déjà acquis est conservé par le client ayant eu son traitement suspendu. ("Pre-emptive Resume Priority" - PRP). Ce modèle de priorité, qui est en fait une généralisation de la politique de service classique PRP, a des applications potentielles en télématique et en télécommunications.

Sous des hypothèses markoviennes quant aux processus des arrivées, nous déterminons la transformée de Laplace-Stieltjes de la loi jointe caractérisant la charge du système à l'état d'équilibre. Ce résultat est obtenu sous forme explicite en réduisant l'analyse à la résolution d'un problèmes aux limites de type Riemann-Hilbert.

Mots clés
Files d'attente ; Processus de Markov ; Equations fonctionnelles ; Problèmes aux limites.

Introduction

Queueing models with state-dependency, connected to the coupling of processors in computer systems, have been extensively studied recently. Analytic methods have been developed by FAYOLLE and IASNOGORODSKI allowing the solution of various coupling problems [5], [6], [10] and important generalizations have been proposed by COHEN and BOXMA [3], both leading to a fairly general methodology in this field [1], [2], [7], [8], [12], [14].

The approach involves the solution of functional equations. Generally the unknown functions are the generating functions of the stationary joint distribution of the number of jobs in the system. This, in turn, leads to the resolution of boundary value problems (Dirichlet, Riemann-Hilbert and Wiener-Hopf problems [9], [13]).

We use a similar machinery to study a queueing system with a single server and two classes of customers, where the service policy is of threshold type. The customers of class 1 [resp. class 2] are queued in buffer 1 [resp. buffer 2] and within classes the customers are served in order of their arrival. The server's attention is given to customer class as follows : when the total amount of work required by class 1 customers is greater than a fixed threshold c, $c \in \overline{\mathbb{R}}^+$, a class 1 customer is always served ; otherwise a class 2 customer is served, and if no class 2 customer is present, a class 1 customer can be served, if any. Customers of both classes can be preempted on the basis of the above description of the service mechanism and a preempted customer will resume his service demand.

Consequently, when $c = 0$ it is seen that this model reduces to the Pre-emptive Resume Priority (PRP) queueing model, (cf.[11]) where the class 1 customers have the priority. Similarly, when $c = +\infty$ this queueing model is the PRP queueing model where the class 2 customers have the priority, which shows that this model is actually a generalization of the PRP queueing system. When $c \in]0,+\infty[$ each class of customers alternatively gets the priority and the allocation of the server to a given class is based on the state of the workload in queue 1 at any time.

On a practical point of view, it is seen that the value of c -the threshold- acts as a control parameter of the system and can therefore be chosen in order to satisfy various constraints required by the users. This can be particularly useful in communication networks when a protocol is used for supporting both time-constrained (e.g. voice, video) and non-time-constrained (e.g. data, facsimile) communication applications. Also this model has potentiel applications in computer systems since in most of them the priority assigned to a job is not fixed and depends, for instance, on the time already spent by the job in the system or on the workload of the system,... ect.

We derive, for Poisson inputs, arbitrary service time distribution for customers of class 2 and exponential service time distribution for customers of class 1, the Laplace-Stieltjes transform of the joint distribution of the workload of the server, at steady state. The case with only exponential service time distributions has been solved in [14] by reducing the analysis to the resolution of a Dirichlet boundary value problem.

We give some definitions and notations in section 1, and we recall in section 2 some basic results which have been obtained in [14]. The sections 3, 4, 5 are devoted to the formulation and the resolution of the boundary value problem and the main result -the L.S.T. transform of the joint distribution of the workload- is stated in section 6. Also explicit formulas are provided for the mean workload in each queue.

1 - The model

We assume that for each class of customers (class 1 and class 2) the input process is a Poisson process with intensity $\lambda_j > 0$, $j=1,2$ respectively. The sequence of service times for each class of customers constitutes a renewal process with an absolutely continuous but otherwise arbitrary renewal distribution, and with finite mean μ_j^{-1}, $j=1,2$. All these processes are assumed to be mutually independent. Within classes the customers are served in order of their arrival. The server's attention is given to each class of customers as follows : when the total amount of work required by class 1 customers is strictly greater than a fixed threshold $c, c \in \overline{\mathbb{R}^+}$, then a class 1 customer is served ; otherwise a class 2 customer is served, and if there are no class 2 customers in the system a class 1 customer is served, if any. This description of the service mechanism necessarily implies that service interruptions can occur for both classes of customers and in that case a preempted customer will resume his service demand. Let us now introduce some notations.

Define for $j = 1,2$ and $\mathrm{Re}\ s \geq 0$:

- $\beta_j(s)$ the Laplace-Stieltjes transform (L.S.T.) of the service time distribution of class j customers ;

- $\gamma_j(s) \stackrel{\text{def}}{=} 1 - \beta_j(s)$;

- $a \stackrel{\text{def}}{=} 1 - \rho_1 - \rho_2$ with $\rho_j \stackrel{\text{def}}{=} \dfrac{\lambda_j}{\mu_j}$;

- $V_j(t) \stackrel{\text{def}}{=}$ the amount of work required by class j customers at time t, including the residual service time of the class j customer in service, if any.

- $\phi^{(n)}(.)$ the n-th derivative of any function $\phi(.)$.

2 - The functional equation

In this section, we briefly recall some basic results which have been obtained in [14].

The stochastic process $\{V_1(t), V_2(t), t > 0\}$ is a Markov process and whenever $a > 0$ it possesses a unique stationary distribution. In the following we shall assume that $a > 0$. Let V_1 and V_2 be two random variables with distribution this stationary distribution, and let us define for $\mathrm{Re}\ x \geq 0$, $\mathrm{Re}\ y \geq 0$

$$H(x,y) \stackrel{\text{def}}{=} E\{e^{-xV_1 - yV_2}\}, \tag{2.1}$$

the L.S.T. of (V_1, V_2).

As a first result, it has been shown in [14] that H(x,y) satisfies the following functional equation for Re x ≧ 0, Re y ≧ 0 :

$$(x-\lambda_1\gamma_1(x)-\lambda_2\gamma_2(y))H(x,y) = (x-y)\ E\{e^{-xV_1-yV_2}(V_1\leq c,\ V_2>0)\}+ax \quad (2.2)$$

where (A) stands for the indicator function of the event {A}.

Let us now introduce $H^-(x,y)$ and $H^+(x,y)$ two bivariate functions defined by

$$H^-(x,y) \overset{def}{=} E\{e^{-x(V_1-c)-yV_2}(V_1\leq c)\} \quad \text{for Re } x\leq 0,\ \text{Re } y\geq 0,$$
$$(2.3)$$
$$H^+(x,y) \overset{def}{=} E\{e^{-x(V_1-c)-yV_2}(V_1> c)\} \quad \text{for Re } x\geq 0,\ \text{Re } y\geq 0.$$

It is readily seen that $H^-(x,y)$ [resp. $H^+(x,y)$] is for fixed y with Re y ≧ 0, an analytic function in x in Re x < 0 [resp. Re x >0], continuous in Re x ≦ 0 [resp. Re x ≧ 0]. Furthermore, for Re x = 0, Re y ≧ 0, the following relation holds

$$H(x,y) = e^{-xc}(H^-(x,y) + H^+(x,y)) , \quad (2.4)$$

which therefore reduces the determination of H(x,y) to the determination of the functions $H^-(x,y)$ and $H^+(x,y)$. To this end and if the service times of class 1 customers are exponentially distributed (i.e. $\gamma_1(s) = s/(\mu_1 + s)$, Re s ≧ 0 which will be assumed from now on), it has been shown in [14] by a routine application of Liouville's theorem, that $H^-(x,y)$ and $H^+(x,y)$ satisfy the following functional equations :

$$\left[\begin{array}{lr} R(x,y)H^-(x,y) = axe^{xc}+(y-x)A(x)+h(x,y)B(y) & \text{for Re } x\leq 0,\ \text{Re } y\geq 0\ ;\ (2.5) \\ S(x,y)H^+(x,y) = -h(x,y)B(y) & \text{for Re } x\geq 0,\ \text{Re } y\geq 0,\ \ (2.6) \end{array}\right.$$

where

B(y) is an unknown analytic function in Re y > 0, continuous in Re y≧0 ;

$$A(x) \overset{\text{def}}{=} E\{e^{-x(V_1-c)}(V_1 \leq c,\ V_2=0\}\ ; \tag{2.7}$$

$$R(x,y) \overset{\text{def}}{=} y - \frac{\lambda_1 x}{\mu_1 + x} - \lambda_2 \gamma_2(y)\ ; \tag{2.8}$$

$$S(x,y) = x-y + R(x,y)\ ; \tag{2.9}$$

$h(x,y) = \dfrac{x - Z(y)}{\mu_1 + x}$ with $Z(y)$ the only root of the equation $S(x,y) = 0$

in Re $x \geqq 0$ for y fixed, with Re $y \geqq 0$. (2.10)

Remark

The assumption that the service times of class 1 customers are exponentially distributed is not essential in this section. Indeed, It is easily seen that the analysis made in [14, section 3] also applies if $\gamma_1(s)$ has a rational form, i.e. $\gamma_1(s) = \gamma_1^1(s)/\gamma_1^2(s)$, where $\gamma_1^i(s)$ is a polynomial in s of degree $n \geqq 1$, $i = 1,2$. In this case the right-hand side of (2.5), (2.6) contain n unknown functions in y, which makes the determination of $H^-(x,y)$ and $H^+(x,y)$ much more complicated than in the case where $n = 1$. For a relevant problem the reader is referred to [3, p319-341]. However, because of the presence of the exponential in the right-hand side of (2.5), the method used in the above mentioned work does not apply here.

3 - The Kernel, its zeros and some preparatory results.

In order to determine the unknown functions $A(x)$ and $B(y)$, cf. eq.(2.5), as the solutions of a boundary value problem according to the method developped in [6], [3], we study the zeros of the "kernel" $R(x,y)$ in the region Re $x \leqq 0$, Re $y \geqq 0$.

The kernel $R(x,y)$ of (2.5) is given by

$$R(x,y) = y - \lambda_2 \gamma_2(y) - \frac{\lambda_1 x}{\mu_1 + x} \text{ for Re } y \geqq 0.$$

Noting the similarities of this kernel with a kernel in [3, p297], let us defined analogously

$$\phi_1(x,w) \stackrel{\text{def}}{=} -\frac{\lambda_1 x}{\mu_1 + x} - w \text{ for } x \in \mathbb{C},$$
$$\phi_2(y,w) \stackrel{\text{def}}{=} y - \lambda_2 \gamma_2(y) + w \text{ for } \operatorname{Re} y \geq 0, \qquad (3.1)$$

for arbitrary w, so that $R(x,y) = \phi_1(x,w) + \phi_2(y,w)$. (3.2)

For all $w \in \mathbb{C}$, $\phi_1(x,w) = 0$ has a unique solution $x = \delta_1(w)$ given by

$$\delta_1(w) = \frac{-\mu_1 w}{\lambda_1 + w}. \qquad (3.3)$$

In the following of this study, the region lying on the left [resp.right] of any smooth contour **U** when moving on this contour in the positive direction will be denoted by $\mathbf{U}^+$ [resp. $\mathbf{U}^-$].

With this notation, it is readily seen from (3.3) that $\operatorname{Re}\delta_1(w) \leq 0$ iff $w \in \mathbf{C}^- \cup \mathbf{C}$, where **C** is the circle with centre $-\lambda_1/2$ and radius $\lambda_1/2$. In particular, $\operatorname{Re}\delta_1(w) \leq 0$ if $\operatorname{Re} w \geq 0$.

On the other hand, it is a well-known result that $\phi_2(y,w)$ has a unique zero $y = \delta_2(w)$ in $\operatorname{Re} y \geq 0$ for $\operatorname{Re} w \leq 0$, $w \neq 0$, and that its multiplicity is one. Since we have assumed that $a > 0$, then $w = 0$ is a zero of multiplicity one of $\phi_2(y,w)$ and moreover $\delta_2(w)$ is analytic in Re $w < 0$ and continuous in $\operatorname{Re} w \leq 0$ [4,p548]. (3.4)

Consequently (3.2) together with the above results show that when $(x,y)=(\delta_1(w), \delta_2(w))$ with $\operatorname{Re} w = 0$ then the kernel $R(x,y)$ vanishes in $\operatorname{Re} x \leq 0$, $\operatorname{Re} y \geq 0$.

This entails from the analycity of the function $H^-(x,y)$ in $\operatorname{Re} x \leq 0$, Re $y \geq 0$, that necessarily the right-hand side of (2.5) must vanish whenever $(x,y) = (\delta_1(w), \delta_2(w))$, with $\operatorname{Re} w = 0$. This reads

$$a\delta_1(w)e^{c\delta_1(w)} + (\delta_2(w)-\delta_1(w))A(\delta_1(w)) + h(\delta_1(w),\delta_2(w))B(\delta_2(w)) = 0,$$

which can be rewritten as

$$(\delta_2(w)-\delta_1(w))(-\Psi(w)+A(\delta_1(w))-G(w)B(\delta_2(w)) = 0 \text{ for Re } w = 0, \tag{3.5}$$

where

$$\Psi(w) \overset{\text{def}}{=} \frac{-a\delta_1(w)e^{c\delta_1(w)}}{\delta_2(w) - \delta_1(w)} \text{ and} \tag{3.6}$$

$$G(w) \overset{\text{def}}{=} \frac{-h(\delta_1(w),\delta_2(w))}{\delta_2(w) - \delta_1(w)}. \tag{3.7}$$

Lemma 3.1

$\delta_1(w) - \delta_2(w)$ has no root for Re $w \leq 0$, $w \neq 0$ and has a root of multiplicity one if $w = 0$.

Proof

For $w \in \mathbf{C}^- \cup \mathbf{C}$, $w \neq 0$, we have seen that Re $\delta_1(w) \leq 0$. On the other hand, Re $\delta_2(w) > 0$ if Re $w \leq 0$, $w \neq 0$. Hence $\delta_1(w) - \delta_2(w) \neq 0$, for all $w \in \mathbf{C}^-\cup\mathbf{C} \cap \{\text{Re } w \leq 0, w \neq 0\}$.

Assume now $w \in \mathbf{C}^+$ and let us show that $\phi_2(\delta_1(w), w) \neq 0$, which will ensure from the definition of $\delta_2(w)$ that $\delta_1(w) \neq \delta_2(w)$. Noting that Re $\delta_1(w) > 0$ for $w \in \mathbf{C}^+$, $\phi_2(\delta_1(w),w)$ can be rewritten as follows :

$$\phi_2(\delta_1(w),w) = \delta_1(w) - \lambda(1-\beta(\delta_1(w))),$$

where $\lambda \overset{\text{def}}{=} \lambda_1+\lambda_2$ and $\beta(s) \overset{\text{def}}{=} \frac{\lambda_1}{\lambda}\beta_1(s)+ \frac{\lambda_2}{\lambda}\beta_2(s)$ for Re $s \geq 0$. The result now follows by applying TAKACS' lemma [16, p49, remark 1].

Differentiating the relation $\delta_2(w)-\lambda_2\gamma_2(\delta_2(w)+w = 0$ as well as (3.3), then letting $w = 0$, yields by a routine calculation $\delta_1^{(1)}(0)-\delta_2^{(1)}(0) = \frac{-a}{(1-\rho_2)\rho_1} \neq 0$, which proves the second statement of the lemma.

□

Consequently, Lemma 3.1 together with (3.5) entail that
$A(\delta_1(w))-G(w)B(\delta_2(w)) = \Psi(w)$ for $\mathrm{Re}\, w = 0,\ w \neq 0.$ (3.8)

From the respective analycity domains of the functions $\delta_1(.)$, $\delta_2(.)$ and $A(.)$, $B(.)$, $h(.)$, cf. section 2, we readily deduce the following results, cf.(3.6), (3.7) and Lemma 3.1 :

$A(\delta_1(w))$ is analytic in $\mathbf{C}^-$ and continuous in $\mathbf{C}^- \cup \mathbf{C}$; (3.9)

$B(\delta_2(w))$ and $G(w)$ are both analytic functions in $\mathrm{Re}\, w < 0$, continuous in $\mathrm{Re}\, w \leq 0$; (3.10)

$\Psi(w)$ is analytic in $\mathbf{C}^- \cap \{\mathrm{Re}\, w < 0\}$, continuous in $\mathbf{C}^- \cup \mathbf{C} \cap \{\mathrm{Re}\, w \leq 0\}$.

(3.11)

Using the continuity of the functions $A(\delta_1(w))$, $G(w)$, $B(\delta_2(w))$ and $\Psi(w)$ on $\mathrm{Re}\, w = 0$, it is seen that relation (3.8) must hold on the entire imaginary axis $\mathrm{Re}\, w = 0$, that is

$$A(\delta_1(w)) - G(w)\, B(\delta_2(w)) = \Psi(w) \quad \text{for } \mathrm{Re}\, w = 0. \tag{3.12}$$

This relation together with (3.9), (3.10), (3.11) will lead in the next section to the formulation of a boundary value problem, which in turn will enable the determination of the two sought functions $A(x)$ and $B(y)$.

Before doing this, let us examine the limits at infinity of $A(\delta_1(w))$ and $G(w)\, B(\delta_2(w))$, which will conclude this preparatory work.

First, it is readily seen, cf.(2.7), (3.3), that

$$\lim_{|w| \to \infty} A(\delta_1(w)) = \alpha \text{ with } 0 < \alpha \overset{\text{def}}{=} A(-\mu_1) < \infty. \tag{3.13}$$

On the other hand, letting x = 0 in (2.6) leads to

$B(y) = \dfrac{-\mu_1\lambda_2(y)\ H^+(0,y)}{Z(y)}$ for Re y ≥ 0, which allows to rewrite the

product $G(w)B(\delta_2(w))$ as follows, cf.(3.6), (3.7) :

$$G(w)B(\delta_2(w)) = \left(\frac{\dfrac{w}{\delta_2(w)}(\mu_1+Z(\delta_2(w))) + \dfrac{\lambda_1}{\delta_2(w)} Z(\delta_2(w))}{\lambda_1(1 + \dfrac{\mu_1}{\lambda_1 + w} \cdot \dfrac{w}{\delta_2(w)})} \right).$$

$$\cdot \left(\frac{-\mu_1\lambda_2\gamma_2(\delta_2(w))\ H^+(0,\delta_2(w))}{Z(\delta_2(w))} \right) \text{ for Re } y \geq 0. \quad (3.14)$$

By definition of $\delta_2(w)$ it is seen that $\delta_2(w)$ satisfies the equation

$$\rho_2\mu_2 \frac{\gamma_2(\delta_2(w))}{\delta_2(w)} = 1 + \frac{w}{\delta_2(w)} \text{ for Re } w \leq 0. \quad (3.15)$$

Noting that $\dfrac{\mu_2\gamma_2(s)}{s}$, Re s ≥ 0, is the L.S.T. of a probability distribution with support $(0,\infty)$, we get from (3.15) that

$$\lim_{\substack{|w| \to \infty \\ \text{Re } w \leq 0}} \frac{w}{\delta_2(w)} = -1. \quad (3.16)$$

Consequently, we readily obtain from (3.14) that

$$\lim_{\substack{|w| \to \infty \\ \text{Re } w \leq 0}} G(w)B(\delta_2(w)) = \beta, \text{ where } 0<\beta \overset{\text{def}}{=} \frac{\lambda_2}{\rho_1}\left(\frac{\mu_1+Z(\infty)}{Z(\infty)}\right)P(V_1>c,\ V_2=0) < \infty, \quad (3.17)$$

where $Z(\infty)$ is finite under the ergodicity condition a > 0 (cf. the definition of Z(y) and [4, p548]).

Let us show that $\alpha = \beta$.

Letting $|w| \to +\infty$ with Re $w = 0$ in (3.12) gives using (3.13), (3.17),

$$\lim_{\substack{|w|\to 0 \\ \text{Re } w = 0}} \Psi(w) = \alpha - \beta.$$

Since $|\Psi(w)| \leq \mu_1 a \dfrac{\left|\dfrac{w}{\delta_2(w)}\right|}{\left|\lambda_1 + w + \mu_1 \dfrac{w}{\delta_2(w)}\right|}$ for $w \in \mathbf{C}^- \cup \mathbf{C}$, it follows using

(3.16) that $\lim\limits_{\substack{|w|\to 0 \\ \text{Re } w = 0}} \Psi(w) = 0$ and therefore $\alpha = \beta$.

Consequently, cf.(3.13), (3.17), $\lim\limits_{|w|\to\infty} A(\delta_1(w)) = \lim\limits_{\substack{|w|\to\infty \\ \text{Re } w \leq 0}} G(w)B(\delta_2(w)) = \alpha.$ (3.18)

4 - Formulation of the boundary value problem

For Re $w = 0$ we have, cf.(3.12),

$$\Psi_1(w) - \Psi_2(w) = \Psi(w), \tag{4.1}$$

where

$$\Psi_1(w) \overset{\text{def}}{=} A(\delta_1(w)) - \alpha \qquad \text{for Re } w \geq 0, \tag{4.2}$$

$$\Psi_2(w) \overset{\text{def}}{=} G(w)B(\delta_2(w)) - \alpha \quad \text{for Re } w \leq 0. \tag{4.3}$$

It is seen that the conditions (3.9)-(3.11), (3.18) and (4.1) formulate a Riemann-Hilbert boundary value problem on the arc Re $w = 0$, cf.[9], [13].

If $\Psi(w)$ satisfies the Hölder condition on Re $w = 0$ (see the remark below) then $\Psi_1(w)$ and $\Psi_2(w)$ are uniquely determined by the following so-called Plemelj-Sokhotski formulas, cf.[9, p25], [13, p42],

$$\Psi_1(w) = \begin{cases} -\dfrac{1}{2i\pi} \displaystyle\int_{-i\infty}^{+i\infty} \dfrac{\Psi(\xi)}{\xi - w}\, d\xi & \text{for Re } w > 0, \qquad (4.4) \\[2ex] \dfrac{1}{2}\Psi(w) - \dfrac{1}{2i\pi} \displaystyle\int_{-i\infty}^{+i\infty} \dfrac{\Psi(\xi)}{\xi - w}\, d\xi & \text{for Re } w = 0\ ; \qquad (4.5) \end{cases}$$

$$\Psi_2(w) = \begin{cases} -\dfrac{1}{2i\pi}\displaystyle\int_{-i\infty}^{+i\infty}\dfrac{\Psi(\xi)}{\xi - w}\,d\xi & \text{for Re } w < 0, \quad (4.6) \\[2ex] -\dfrac{1}{2}\Psi(w) - \dfrac{1}{2i\pi}\displaystyle\int_{-i\infty}^{+i\infty}\dfrac{\Psi(\xi)}{\xi - w}\,d\xi & \text{for Re } w = 0. \quad (4.7) \end{cases}$$

Note that the singular Cauchy integrals in (4.5), (4.7) are defined by their principal value (cf.[4], p19).

Remark

For Re $w = 0$, $\Psi(w)$ possesses a derivative and it therefore satisfies the Hölder condition on this arc (cf.[9], p6).

Indeed, we get from (3.6),

$$\Psi^{(1)}(w) = \frac{a\mu_1 e^{c\delta_1(w)}}{(\lambda_1 + w)^2}\,.$$

$$\cdot\frac{[(1+cw\delta_1^{(1)}(w))(\delta_2(w)(\lambda_1+w)+\mu_1 w)-w(\delta_2^{(1)}(w)(\lambda_1+w)+\delta_2(w)+\mu_1)]}{[\delta_2(w) - \delta_1(w)]^2}\,.$$

Using (3.16) it is seen that $\lim\limits_{|w|\to\infty} \Psi^{(1)}(w)$ is finite. It remains to check that $\Psi^{(1)}(0)$ is well-defined, since from Lemma 3.1 the denominator of $\Psi^{(1)}(w)$ only vanishes on Re $w = 0$ for $w = 0$. A standard calculation then gives

$$\Psi^{(1)}(0) = \frac{2(1-\rho_2)\left(1-\dfrac{ac\mu_1^2}{\rho_1}\right)+\dfrac{\lambda_1\lambda_2\beta_2^{(2)}(0)}{1-\rho_2}}{2\mu_1 a},$$ which is defined under

the ergodicity condition $a > 0$.

5 - Analytic continuation

In the previous section, $A(\delta_1(w))$ for Re $w \geq 0$ and $B(\delta_2(w))$ for Re $w \geq 0$ have been obtained as the solutions of a boundary value problem, up to a positive constant α.

However in order to completely determine $A(x)$ in Re $x \leq 0$ and $B(y)$ in Re $y \geq 0$, the inverse functions of $\delta_i(w)$, $i = 1,2$ have to be investigated.

Because of (3.4), the conformal mapping

$w \to y = \delta_2(w)$ of Re $w \leq 0$ into Re $y \geq 0$, has a unique inverse $w = \lambda_2 \gamma_2(y) - y$ so that $\delta_2(0) = 0$. (5.1)

This implies that relations (4.6), (4.7) uniquely determine $B(y)$ for Re $y \geq 0$.

Similarly, the mapping

$w \to x = \delta_1(w)$ of $\mathbf{C}^- \cup \mathbf{C}$ into Re $x \leq 0$ has a unique inverse

$$w = \frac{-\lambda_1 x}{\mu_1 + x}. \tag{5.2}$$

Consequently, the function $A(\delta_1(w))$ which is only given in Re $w \geq 0$ by eqs.(4.4), (4.5), must be analytically continued to $\mathbf{C}^- \cup \mathbf{C}$, in order to get $A(x)$ in the whole region Re $x \leq 0$.

This is done in the following

Lemma 5.1

The analytic continuation of $A(\delta_1(w))$ to $\mathbf{C}^- \cup \mathbf{C}$ is given by

$$F(w) \overset{\text{def}}{=} \alpha + \Psi(w) - \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} \frac{\Psi(\xi)}{\xi - w} d\xi \quad \text{for } w \in \mathbf{D} \cup \mathbf{C} / \{0\}, \tag{5.3}$$

where

$$D \overset{\text{def}}{=} \mathbf{C}^{-} \cap \{\mathrm{Re}\, w < 0\} .$$

Proof

The analycity and the continuity of the right-hand side of (5.3) respectively in $\mathbf{D}$ and in $\mathbf{D} \cup \mathbf{C} \cup \{\mathrm{Re}\, w = 0,\ w \neq 0\}$ follow from (3.11) and the remark of section 4.

On the other hand, since $\Psi(w)$ satisfies the Hölder condition on $\mathrm{Re}\, w = 0$, we get by applying the Plemelj-Sokhotski formula

$$\lim_{\substack{w \to z \\ \mathrm{Re}\, z = 0 \\ w \in \mathbf{D}}} \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} \frac{\Psi(\xi)}{\xi - w} d\xi = \frac{1}{2} \Psi(z) + \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} \frac{\Psi(\xi)}{\xi - z} d\xi, \qquad (5.4)$$

where the singular Cauchy integral in the right-hand side is defined by its principal value.

Hence, from (5.3), (5.4) it is seen that

$$\lim_{\substack{w \to z \\ \mathrm{Re}\, z = 0 \\ w \in \mathbf{D}}} F(w) = \alpha + \frac{1}{2} \Psi(z) - \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} \frac{\Psi(\xi)}{\xi - z} d\xi = A(\delta_1(z)) ,$$

which shows from the principle of the analytic continuation, that $F(w)$ is the analytic continuation of $A(\delta_1(w))$ to $\mathbf{C}^{-} \cup \mathbf{C}$. $\square$

In particular this result enables the determination of the constant α. Indeed, we have from (2.7)

$$\lim_{\substack{x \to -\infty \\ x \in \mathbb{R}}} A(x) = \left[\begin{array}{ll} 0 & \text{if} \quad c > 0, \\ a & \text{if} \quad c = 0. \end{array} \right. \qquad (5.5)$$

Furthermore, since $\lim\limits_{\substack{w\to-\lambda_1 \\ w\in]-\infty,-\lambda_1[}} \delta_1(w) = -\infty$, we get using (3.6), (5.3) and (5.5) that

$$\alpha = \frac{1}{2i\pi}\int_{-i\infty}^{+i\infty} \frac{\Psi(\xi)}{\xi+\lambda_1}\, d\xi \quad \text{for } c \geq 0, \tag{5.6}$$

since

$$\lim_{\substack{w\to-\lambda_1 \\ w\in]-\infty,-\lambda_1[}} \Psi(w) = \left[\begin{array}{ll} 0 & \text{if } c > 0, \\ a & \text{if } c = 0. \end{array}\right.$$

By introducing the inverse mappings of $\delta_1(w)$ and $\delta_2(w)$ into eqs.(4.4)-(4.7), it is readily obtained together with Lemma 5.1, (4.2), (4.3) and (5.6) that

$$A(x) = \left[\begin{array}{lll} A_1(x) + I(x) & \text{for } x \in L^- \cap \{\mathrm{Re}\, x \leq 0\}, & (5.7) \\ \frac{1}{2}A_1(x) + I(x) & \text{for } x \in L, & (5.8) \\ I(x) & \text{for } x \in L^+; & (5.9) \end{array}\right.$$

$$B(y) = \left[\begin{array}{lll} J(y) & \text{for } \mathrm{Re}\, y \geq 0,\ \mathrm{Re}(\lambda_2\gamma_2(y)-y) < 0, & (5.10) \\ B_1(y) + J(y) & \text{for } \mathrm{Re}\, y \geq 0,\ \mathrm{Re}(\lambda_2\gamma_2(y)-y) = 0, & (5.11) \end{array}\right.$$

where L is the circle with centre $-\frac{\mu_1}{2}$ and radius $\frac{\mu_1}{2}$;

$$A_1(x) \overset{\text{def}}{=} -\frac{axe^{cx}}{\delta_2\left(\frac{-\lambda_1 x}{\mu_1+x}\right)-x}; \tag{5.12}$$

$$B_1(y) \overset{\text{def}}{=} - \frac{\Psi(\lambda_2\gamma_2(y)-y)}{2G(\lambda_2\gamma_2(y)-y)} \quad ; \tag{5.13}$$

$$I(x) \overset{\text{def}}{=} - \frac{\lambda_1\mu_1}{2i\pi} \int_{-i\infty}^{+i\infty} \frac{\Psi(\xi)}{[\xi(\mu_1+x)+\lambda_1 x][\xi+\lambda_1]} d\xi \ ; \tag{5.14}$$

$$J(y) \overset{\text{def}}{=} - \frac{\lambda_1+\lambda_2\gamma_2(y)-y}{2i\pi G(\lambda_2\gamma_2(y)-y)} \int_{-i\infty}^{+i\infty} \frac{\Psi(\xi)}{[\xi+y-\lambda_2\gamma_2(y)][\xi+\lambda_1]} d\xi . \tag{5.15}$$

$I(x)$ for $x \in \mathbb{C}$ and $J(y)$ for $\operatorname{Re} y \geq 0$ are computed in the appendix.

6 - L.S.T. of the joint distribution of (V_1, V_2)

Theorem 6.1 (L.S.T. of the joint distribution of the workload)

The Laplace-Stieltjes transform $H(x,y)$ of the distribution of (V_1,V_2) is given for $c \geq 0$, $a > 0$ by :

$$H(x,y)=\frac{1}{R(x,y)} \left[ax+(y-x)\left(A(x)- \frac{(x-Z(y))B(y)}{(\mu_1+x)S(x,y)}\right)e^{-xc}\right], \ \operatorname{Re} x \geq 0, \ \operatorname{Re} y \geq 0, \tag{6.1}$$

where

$e^{-xc}A(x) = \left(\mathbf{1}_{\{x\neq0\}} + \frac{1}{2}\mathbf{1}_{\{x=0\}}\right) A_1(x) + I(x)$ for $\operatorname{Re} x \geq 0$ where $A_1(x)$

and $I(x)$ are respectively given by (5.12), (5.14) ;

$B(y)$ is given by (5.10), (5.11) ;

$$R(x,y) = y - \frac{\lambda_1 x}{\mu_1+x} - \lambda_2\gamma_2(y) \ ;$$

$$S(x,y) = x-y + R(x,y) \qquad ;$$

$Z(y)$ is the unique root in $\operatorname{Re} x \geq 0$ of the equation $S(x,y) = 0$, for $\operatorname{Re} y \geq 0$.

Proof

For Re $x = 0$, Re $y \geq 0$, we have from (2.4), (2.5), (2.6),

$$H(x,y) = \frac{1}{R(x,y)}[ax+(y-x)A(x)e^{-xc}+h(x,y)B(y)e^{-xc}] - \frac{h(x,y)}{S(x,y)}B(y)e^{-xc}.$$

A simple calculation then gives the relation (6.1). (6.2)

Let us show that the right-hand side of (6.2) can be analytically continued to Re $x \geq 0$, Re $y \geq 0$. From the definitions of the functions R, S, h, cf.(2.8), (2.9), (2.10), it is sufficient to show that the term between square brackets in the right-hand side of (6.2) can be analytically continued to Re $x \geq 0$, Re $y \geq 0$, and that its analytic continuation vanishes whenever $R(x,y)$ vanishes in Re $x \geq 0$, Re $y \geq 0$.

The analytic continuation of the function $e^{-xc}A(x)$ in Re $x \geq 0$ is given by (5.7), that is

$$e^{-xc}A(x) = \frac{-ax}{\delta_2\left(\frac{-\lambda_1 x}{\mu_1+x}\right)-x} + e^{-xc}\,I(x) \quad \text{for Re } x \geq 0. \tag{6.3}$$

This result follows from the principle of analytic continuation since the right-hand side of (6.3) is analytic in Re $x > 0$ and continuous in Re $x \geq 0$ (cf.(3.4), (5.14) and Lemma 3.1).

Consequently, (6.3) together with the analycity of $h(x,y)$ in Re $x \geq 0$, Re $y \geq 0$ gives the analytic continuation in Re $x \geq 0$, Re $y \geq 0$, of the term between square brackets in (6.2).

Let us define

$$p(x,y) \overset{\text{def}}{=} ax+(y-x)A(x)e^{-xc}+h(x,y)B(y)e^{-xc} \quad \text{for Re } x \geq 0,\ \text{Re } y \geq 0, \tag{6.4}$$

where $A(x)e^{-xc}$ is given by (6.3).

It remains to check that $p(x,y)$ vanishes whenever $R(x,y)$ vanishes in Re $x \geq 0$, Re $y \geq 0$.

As a first result, we have that for fixed x with $\mathrm{Re}\, x \geq 0$, the equation $R(x,y) = 0$ has a unique root $y = y(x)$ in $\mathrm{Re}\, y \geq 0$ [4, p.548]. Let (x_0,y_0) be a zero of $R(x,y)$ with $\mathrm{Re}\, x_0 \geq 0$, $\mathrm{Re}\, y_0 \geq 0$, $y_0 \overset{\mathrm{def}}{=} y(x_0)$, and let us show that $p(x_0,y_0) = 0$, which will conclude the proof.

Multiplying (3.5) by $e^{-c\delta_1(w)}$ we get from (3.6), (3.7), (6.4),

$$p(\delta_1(w), \delta_2(w)) = 0 \text{ for } \mathrm{Re}\, w = 0. \tag{6.5}$$

From (6.4) and the principle of permanence [15, p.107] it is seen that relation (6.5) also holds for $w \in \mathbf{C}^+ \cup \mathbf{C}$ (see the definition of $\mathbf{C}$ in section 3) since for such w we have $\mathrm{Re}\, \delta_1(w) \geq 0$, $\mathrm{Re}\, \delta_2(w) \geq 0$.

Let us define $w_0 \overset{\mathrm{def}}{=} \dfrac{-\lambda_1 x_0}{\mu_1 + x_0}$ $(= -(y_0 - \lambda_2\gamma_2(y_0))$.

Since $w_0 \in \mathbf{C}^+ \cup \mathbf{C}$ (cf.(5.2)) and that $(x_0,y_0) = (\delta_1(w_0),\delta_2(w_0))$ by definition of $\delta_1(w)$ and $\delta_2(w)$, we deduce by applying (6.5) with $w \in \mathbf{C}^+ \cup \mathbf{C}$ that $p(x_0,y_0) = 0$.

$\square$

The previous theorem enables the determination of all the moments of V_1 and V_2. For instance, let us consider $E\{V_1\} = -\frac{\partial}{\partial x} H(x,0)\,|_{x=0}$, the mean workload in queue 1.

From (6.1) we immediately get

$$E\{V_1\} = -\frac{1}{\lambda_1}\{A(0)(1-\mu_1 c)+\mu_1 A^{(1)}(0)-a+B^{(1)}(0)[\frac{c\mu_1}{\mu_1-\lambda_1} + \frac{\lambda_1}{(\mu_1-\lambda_1)^2}]\}. \tag{6.6}$$

On the other hand, letting $w = 0$ in (3.12) gives

$$B(0) = \frac{A(0)-\Psi(0)}{G(0)},$$

where after a routine calculation we get (cf.(3.6), (3.7)),

$$G(0) = \frac{1}{\mu_1 - \lambda_1} \text{ and } \Psi(0) = 1 - \rho_2.$$

By putting these results into (6.6), it follows that

$$E\{V_1\} = -\frac{1}{\rho_1}\left(\frac{A(0) - \rho_1(1-\rho_2)}{\mu_1(1-\rho_1)} + A^{(1)}(0) - \frac{a}{\mu_1} - (1-\rho_2)c\right). \tag{6.7}$$

Using now (5.8), (5.12), (5.14), a tedious but routine calculation gives

$$A(0) = \frac{1-\rho_2}{2} - \lambda_1 K(0) \tag{6.8}$$

and

$$A^{(1)}(0) = \frac{(1-\rho_2)}{2a}\left[ac - \frac{\rho_1}{\mu_1} - \frac{\rho_1^2 \lambda_2 \beta_2^{(2)}(0)}{2(1-\rho_2)^2}\right] + \rho_1 \left[K(0)+\lambda_1 K^{(1)}(0)\right], \tag{6.9}$$

where $K(w) \overset{\mathrm{def}}{=} \frac{1}{2i\pi}\int_{-i\infty}^{+i\infty} \frac{\Psi(\xi)}{(\lambda_1+\xi)(\xi-w)}\, d\xi$ has been computed in the appendix, which completely determines $E\{V_1\}$.

$E\{V_2\}$, the mean workload in queue 2, can now be directly obtained from (6.7) by using the following argument : at steady state, the total workload in the system is obviously equal to the stationary workload of an M/G/1 queue with input parameter λ and with L.S.T. of the service time distribution $\beta(.)$, (see the definitions of λ and $\beta(.)$ in the proof of the Lemma 3.1).

Consequently, $E\{V_1 + V_2\} = \dfrac{2\dfrac{\lambda_1}{\mu_1^2} + \lambda_2 \beta_2^{(2)}(0)}{2a}$, (cf.[16, p68])

which allows together with (6.7)-(6.9) the determination of $E\{V_2\}$.

As a verification, let us consider the case $c = 0$, for which we know that $E\{V_1\} = \rho_1/(\mu_1(1-\rho_1))$, since in this case the class 1 customers have the same behavior as the customers of an M/M/1 queue.

From relations (A.9), (A.10) of the appendix, it is readily seen that

for $c = 0$, $K(0) = \dfrac{-a}{\lambda_1} + \dfrac{1-\rho_2}{2\lambda_1}$ and

$$K^{(1)}(0) = \frac{1}{\lambda_1^2}\left[a + \left(\frac{1-\rho_2}{2}\right)\left(\frac{\rho_1 - a}{a}\right) + \frac{\mu_1 \rho_1^2 \lambda_2 \beta_2^{(2)}(0)}{4a(1-\rho_2)}\right].$$

Substituting these results into (6.8), (6.9), we get $A(0) = a$ and $A^{(1)}(0) = 0$, which were the expected results on account of (2.7).

Finally by introducing these values into (6.7) it is seen that

$$E\{V_1\} = \frac{\rho_1}{\mu_1(1-\rho_1)} .$$

When $c > 0$, the constants $K(0)$ and $K^{(1)}(0)$ involved in the expression of $E\{V_1\}$ can be computed using a simple numerical procedure.

Conclusion

A two-queue system with a single server, two classes of customers and where the service policy is of threshold type has been considered. Under particular probabilistic assumptions -Poisson inputs, exponential and arbitrary service time distributions- we have obtained the L.S.T. of the joint distribution of the workload in the system by solving a boundary value problem.

The more general case with only arbitrary service time distributions remains unsolved, and leads to a functional equation (eq.(2.2)) of a non-standard type (i.e. which cannot be reduced to those encountered in [3], [5] and [10]) which deserves further research investigations.

I should like to thank G. Fayolle for suggesting this research topic and for many helpful discussions during the course of this work.

APPENDIX : Computation of $I(x)$ and $J(y)$.

For $w \in \mathbb{C}$, let us define

$$K(w) \overset{\text{def}}{=} \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} \frac{\Phi(\xi)}{\xi - w} d\xi, \text{ where } \Phi(\xi) \overset{\text{def}}{=} \frac{\Psi(\xi)}{\xi + \lambda_1} . \tag{A.1}$$

From (5.14), (5.15) it is seen that $I(x)$ and $J(y)$ can be expressed in terms of $K(.)$ as follows :

$$I(x) = -\left(\frac{\mu_1\lambda_1}{\mu_1+x}\right) K\left(\frac{-\lambda_1 x}{\mu_1+x}\right), \ x \in \mathbb{C}, \tag{A.2}$$

$$J(y) = (y-\lambda_2\gamma_2(y)-\lambda_1)\ K(\lambda_2\gamma_2(y)-y) \ / \ G(\lambda_2\gamma_2(y)-y), \ \text{Re } y \geq 0. \tag{A.3}$$

The following of this appendix is devoted to the determination of $K(.)$.

Let us define (cf. Figure 1)

$\mathbf{C}(R)$ the semi-circumference contained in the left-half complex plane, with centre 0 and radius R,

$\Delta(R) = \mathbf{C}(R) \cup [-iR, iR]$,

$\mathbf{C}_w(\varepsilon)$ for Re $w = 0$ the semi-circumference contained in the left-half complex plane with centre w and radius ε,

$\Delta_w(\varepsilon,R) = \mathbf{C}(R) \cup [-iR, w-i\varepsilon] \cup \mathbf{C}_w(\varepsilon) \cup [w+i\varepsilon, R]$, for large R and small ε.

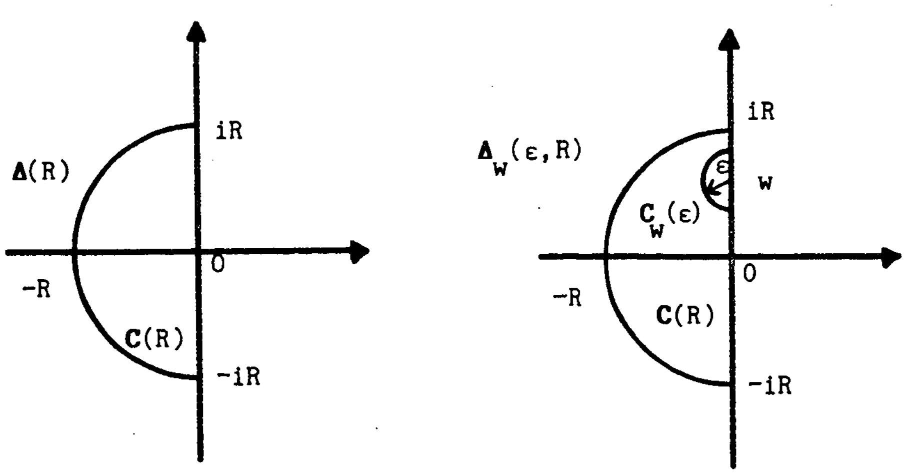

Figure 1 : The contours $\Delta(R)$ and $\Delta_w(\varepsilon,R)$

Lemma A.1

$$K(w) = \lim_{R \to +\infty} \frac{1}{2i\pi} \int_{\Delta(R)} \frac{\Phi(\xi)}{\xi - w} d\xi, \quad w \in \mathbb{C} .$$

Proof

$$\int_{\Delta(R)} \frac{\Phi(\xi)}{\xi-w} d\xi = \int_{C(R)} \frac{\Phi(\xi)}{\xi-w} + \int_{-iR}^{+iR} \frac{\Phi(\xi)}{\xi-w} d\xi.$$

For $\xi \in C(R)$, we deduce from (5.14) that $\Phi(\xi) \to 0$ uniformly with respect to arg ξ whenever $R \to +\infty$.

Consequently, $\left| \int_{C(R)} \frac{\Phi(\xi)}{\xi-w} d\xi \right| \leq \frac{\pi R}{| R - |w| |} \max_{\xi \in C(R)} | \Phi(\xi) |$ and

$$\lim_{R \to +\infty} \frac{1}{2i\pi} \int_{\Delta(R)} \frac{\Phi(\xi)}{\xi-w} d\xi = K(w).$$ □

For the computation of $K(w)$, $w \in \mathbb{C}$, we distinguish the two following cases :

1) Re $w \neq 0$

For $\xi \in \Delta(R)^+$ and sufficiently large R, it is readily seen, cf.(5.2), and Lemma 5.1 that $\Phi(\xi)$ is regular, except for $\xi = -\lambda_1$, where it has an essential singular point.

Applying the Cauchy's residue theorem, we get using Lemma A.1,

$$K(w) = \lim_{R \to +\infty} \frac{1}{2i\pi} \int_{\Delta(R)} \frac{\Phi(\xi)}{\xi-w} d\xi = \text{Res} \left(\frac{\Phi(\xi)}{\xi-w} ; -\lambda_1 \right)$$

$$+ \mathbf{1}_{\{\text{Re } w < 0,\ w \neq -\lambda_1\}} \text{Res} \left(\frac{\Phi(\xi)}{\xi-w} ; w \right) , \qquad (A.4)$$

where Res $(\delta(\xi);\gamma)$ denotes the residue of $\delta(\xi)$ at point $\xi = \gamma$, and where $\mathbf{1}_{\{A\}}$ is the indicator function of the event $\{A\}$.

Define $R_1(w) \overset{\text{def}}{=} \text{Res} \left(\frac{\Phi(\xi)}{\xi-w} ; -\lambda_1 \right)$, $w \in \mathbb{C}$.

A routine calculation yields,

$$R_1(w) = \begin{cases} e^{-c\mu_1} \sum\limits_{p=0}^{+\infty} \dfrac{(\lambda_1 c\mu_1)^p}{(p!)^2} F_{(w)}^{(p)}(-\lambda_1) & \text{if } w \neq -\lambda_1, \\ e^{-c\mu_1} \sum\limits_{p=0}^{+\infty} \dfrac{(\lambda_1\mu_1 c)^p}{p!(p+1)!} & \tilde{F}^{(p+1)}(-\lambda_1) \text{ if } w = -\lambda_1, \end{cases} \tag{A.5}$$

where

$$F_w(\xi) \overset{\text{def}}{=} \frac{a\mu_1\xi}{(\xi-w)[\delta_2(\xi)(\lambda_1+\xi)+\mu_1\xi]} \ ;$$

$$\tilde{F}(\xi) \overset{\text{def}}{=} \frac{a\mu_1\xi}{\delta_2(\xi)(\lambda_1+\xi)+\mu_1\xi} .$$

For $w \neq -\lambda_1$, we immediately get Res $\left(\dfrac{\Phi(\xi)}{\xi - w} \ ; w\right) = \Phi(w)$. (A.6)

Hence, from (A.4) - (A.6) we get for Re $w \neq 0$

$$K(w) = R_1(w) + 1_{\{Re w<0, w\neq -\lambda_1\}} \Phi(w). \tag{A.7}$$

2) Re $w = 0$

In that case, $K(w)$ is a singular integral which is defined by its principal value.

Let us now consider the contour $\Delta_w(\varepsilon,R)$ for sufficiently large R and small ε.

Since $\lim\limits_{\substack{\varepsilon \to 0 \\ \xi \in \mathbf{C}_w(\varepsilon)}} \Phi(\xi) = \Phi(w)$, it is well-known that

$$\lim_{\varepsilon \to 0} \frac{1}{2i\pi} \int_{C_w(\varepsilon)} \frac{\Phi(\xi)}{\xi - w} d\xi = \frac{-\Phi(w)}{2}$$, where ξ moves on $C_w(\xi)$ in the negative direction.

Similarly to Lemma A.1, it can be shown that :

$$K(w) = \lim_{\substack{R \to +\infty \\ \varepsilon \to 0}} \frac{1}{2i\pi} \int_{\Delta_w(\varepsilon,R)} \frac{\Phi(\xi)}{\xi-w} d\xi - \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \int_{C_w(\varepsilon)} \frac{\Phi(\xi)}{\xi-w} d\xi,$$

Hence it is obtained that $K(w) = R_1(w) + \frac{\Phi(w)}{2}$ for Re $w = 0$, using the Cauchy's residue theorem. (A.8)

Finally, it comes from (A.1), (A.7), (A.8),

$$\boxed{K(w) = R_1(w) + \frac{\Psi(w)}{\lambda_1 + w}\left[1_{\{\mathrm{Re}\, w<0,\, w\neq-\lambda_1\}} + \frac{1}{2}\, 1_{\{\mathrm{Re}\, w = 0\}}\right]} \quad ,(A.9)$$

where R_1 is given by (A.5).

<u>Remark</u>

If $c = 0$, we immediately get from A.5,

$$R_1(w) = \begin{cases} \dfrac{-a}{(\lambda_1+w)} & \text{if } w \neq -\lambda_1, \\[2ex] \dfrac{a\delta_2(-\lambda_1)}{\lambda_1\mu_1} & \text{if } w = -\lambda_1. \end{cases} \quad (A.10)$$

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